

INTRODUCTION TO GENERALIZED CATEGORIES

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ABSTRACT. We present a formal introduction to the theory of generalized categories [14]. We describe functors, equivalences, natural transformations, adjoints, and limits in the generalized setting.

1. INTRODUCTION

Category theory [2, 8, 11] has its origins in mathematics, and has since become a well-established area of foundations, with a rich interaction with computer science. It begins with the insight that diagrams and morphisms have a mathematical life unto themselves, independent of function theory, and independent of any use of points as arguments. In [5], the paper on natural transformations in which the elementary notions of category theory are introduced for the first time, Eilenberg and MacLane write

It is thus clear that the objects play a secondary role, and could be entirely omitted from the definition of a category. However, the manipulation of the applications would be slightly less convenient were this done.

Thus two views have been known to category theorists since the beginning of the subject: a one-sorted definition describing a universe of pure maps, and a two-sorted definition including the objects that, in applications, are prior to the maps that they inspire. In practice, these two views pull against one another in a way that seems perhaps like a natural, irresolvable tension. The latter approach has proven to be the dominant one, while the former approach has made occasional appearances, for example in work by Ehresmann [4], Street [17], and in recent work by Cockett [3].

The two-sorted view masks the potential for generalization that begins with the less-often-used one-sorted formulation. In the latter case, an axiom requires that the source and target maps s and t are trivial upon iteration: $ss = st = s$, $tt = ts = t$. This condition, however, is extraneous—it never arises in proofs. Dropping it gives rise to a rather general notion, which may be weakened further via replacing some equalities with inequalities, as suggested by some kinds of applications [13, 15]. To the best of our knowledge there exists no prior investigation of this generalization of category theory. Having noticed this deficit in the literature along with a few signs that such a theory might be sufficiently strong to have potential applications, the author made an investigation [14]. A promising direction for applications of the theory appears to lay in computer science, more specifically programming language theory, where it offers promise as a usable semantics for higher-kinded types, metaprogramming, dependent type theory, and other generalized forms of type theory (see, for example, [1]). In [14] the theory of categories and categorical logic is developed in the generalized setting, including parts of topos theory.

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This short article is an introduction to the basic generalized category theory used in [14]: the theory of functors, equivalences, natural transformations, adjoints, and limits in the generalized setting.

2. GENERALIZED CATEGORIES

Preliminaries. We write composition $G \circ F := (f \mapsto G(F(f)))$ and in general, for mappings F and G with common domain and codomain (in which concatenation is meaningful) we define the operation

$$G \triangle F := (f \mapsto G(f)F(f)),$$

the standard vertical composition operation [11]. In any context where it is meaningful, we use the standard arrow notation $f : a \rightarrow b$ to mean that an element f is given, the source of f is a , and the target of f is b . The notation \downarrow indicates that all composed pairs of elements in the expression or relation are in fact composable pairs.

2.1. Definition.

Definition 1. A *generalized category* is a structure $(\mathcal{C}, \sqsubseteq, \mathbf{s}, \mathbf{t}, \cdot)$ where \mathcal{C} is a set, \sqsubseteq is a relation on \mathcal{C} , \mathbf{s} and \mathbf{t} are mappings $\mathcal{C} \rightarrow \mathcal{C}$, and (\cdot) is a partially defined mapping $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, denoted $a \cdot b$ or ab . These are required to satisfy

- (1) $(\mathcal{C}, \sqsubseteq)$ is a partially ordered set,
- (2) $ab \downarrow$ if and only if $\mathbf{s}(a) \sqsubseteq \mathbf{t}(b)$.
- (3) If $(ab)c \downarrow$ or $a(bc) \downarrow$ then $(ab)c = a(bc)$.
- (4) If $ab \downarrow$ then $\mathbf{s}(ab) = \mathbf{s}(b)$ and $\mathbf{t}(ab) = \mathbf{t}(a)$.
- (5) (Element-Identity) For all $a \in \mathcal{C}$, there exists $b \in \mathcal{C}$ such that
 - (a) $\mathbf{s}(b) = \mathbf{t}(b) = a$,
 - (b) if $bc \downarrow$ then $bc = c$,
 - (c) if $cb \downarrow$ then $cb = c$,
- (6) (Object-Identity) Let $a \in \mathcal{C}$ and $\mathbf{s}(a) = \mathbf{t}(a) = a$. Then
 - (a) if $ba \downarrow$ then $ba = b$.
 - (b) If $ab \downarrow$ then $ab = b$.
- (7) (Order Congruences¹)
 - (a) If $a \sqsubseteq b$ then $\mathbf{s}(a) \sqsubseteq \mathbf{s}(b)$ and $\mathbf{t}(a) \sqsubseteq \mathbf{t}(b)$.
 - (b) $a \sqsubseteq b$ and $c \sqsubseteq d$ and $ac, bd \downarrow$ implies $ac \sqsubseteq bd$.
 - (c) $a \sqsubseteq b$ implies $1_a \sqsubseteq 1_b$.

The element c of axiom (5) is unique, and is denoted 1_a or id_a , and called the *identity* on a .

As a partially ordered set a generalized category resembles, but is weaker than, a domain [7], indeed motivation for the ordering comes from domain theory [15, 18]. If $a \sqsubseteq b$, we say that a *approximates* b , and b *sharpens* a . When the ordering \sqsubseteq is nontrivial, one may call \mathcal{C} a *proximal* generalized category. We often think of proximal categories as having at least a bottom element \perp , but we do not assume this in the definition, since we would like, as a special case, for an ordinary one-category to be a generalized category. If the order given by \sqsubseteq is discrete, we might say that the generalized category is *discrete*, and similarly for other order-theoretic attributes, but as this may lead to confusion with the notion of a discrete category (one with essentially no morphisms), we shall say instead that such a generalized category is a *sharp* generalized category. We allow

¹These axioms are needed for the Kleisli construction [14].

ourselves to refer to a *proximal* generalized category whenever we wish to emphasize that we refer to a generalized category that is not assumed to be sharp.

An *element* $f \in \mathcal{C}$ is an element f of the underlying set \mathcal{C} . An *object* a in \mathcal{C} is an element a of \mathcal{C} such that $\mathbf{s}(a) = \mathbf{t}(a) = a$. We write $\text{Ob}(\mathcal{C})$ for the set of objects. For $a \in \mathcal{C}$, we define the *height* of a , denoted $\text{height}(a)$, to be the maximum of the set of nonnegative integers n such that there exists a sequence \vec{s} of source and target operations of length n such that $\vec{s}(i)$ is an object, unless there is an infinite sequence \vec{s} of source and target operations such that no subsequence yields an object. In that case, we say that $\text{height}(a) = \infty$.

With this terminology, Definition 1 says that in a generalized category with identities, every element a has an identity 1_a , and that if the element is an object, this identity is a itself. If $a \in \mathcal{C}$ has identity 1_a and is not an object, then $a \neq 1_a$.

The maps \mathbf{s} and \mathbf{t} of the definition are called the *source* or *domain* and *target* or *codomain* maps, respectively. We may sometimes denote the map $\mathbf{s}(a)$ by $\text{dom}(a)$ or \bar{a} , and the map $\mathbf{t}(a)$ by $\text{cod}(a)$ or \hat{a} .

Given a generalized category \mathcal{C} , any element of \mathcal{C} may be composed with other compatible elements, and it is equipped with a “tail” of fellow elements, defined by the \mathbf{s} and \mathbf{t} maps. We think of the product as developing from right to left, and we may write $c : a \rightarrow b$ when $\mathbf{s}(a) = b$, $\mathbf{t}(a) = c$. Note as an aside that if one pictures instead a representation $a = {}_c a_b$ of a , one has a picture of composition ${}_c a_b {}_d e = {}_c (ad)_e$. This notation can be iterated to

$$a = {}_g c_f a_e b_d$$

In this manner one can visualize a binary tree.

2.2. An Alternative Approach. We pause to make note of an alternative approach, and discuss why we choose the approach of Definition 1.

Definition 2. A *generalized category* is a structure $(\mathcal{C}, \sqsubseteq, \mathbf{s}, \mathbf{t}, \cdot)$ where \mathcal{C} is a set, \sqsubseteq is a relation on \mathcal{C} , \mathbf{s} and \mathbf{t} are operators (mappings) $\mathcal{C} \rightarrow \mathcal{C}$, and (\cdot) is a partially defined binary operation $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, denoted $a \cdot b$ or ab . These are required to satisfy

- (1) $(\mathcal{C}, \sqsubseteq)$ is a partially ordered set,
- (2) If $(ab)c \downarrow$ or $a(bc) \downarrow$ then $(ab)c = a(bc)$.
- (3) If $ab \downarrow$ then $\mathbf{s}(ab) = \mathbf{s}(b)$ and $\mathbf{t}(ab) = \mathbf{t}(a)$.
- (4) $ab \downarrow$ if and only if $\mathbf{s}(a) \sqsubseteq \mathbf{t}(b)$.
- (5) (Object-Identity) Let $a \in \mathcal{C}$ and $\mathbf{s}(a) = \mathbf{t}(a) = a$. Then
 - (a) if $ba \downarrow$ then $ba = b$.
 - (b) If $ab \downarrow$ then $ab = b$.
- (6) (Order Congruences)
 - (a) If $a \sqsubseteq b$ then $\mathbf{s}(a) \sqsubseteq \mathbf{s}(b)$ and $\mathbf{t}(a) \sqsubseteq \mathbf{t}(b)$.
 - (b) $a \sqsubseteq b$ and $c \sqsubseteq d$ and $ac, bd \downarrow$ implies $ac \sqsubseteq bd$.
 - (c) $a \sqsubseteq b$ implies $1_a \sqsubseteq 1_b$.

A generalized category is said to be equipped *with identities* if for every $a \in \mathcal{C}$, if there exists $b \in \mathcal{C}$ such that $\mathbf{s}(b) = a$ or $\mathbf{t}(b) = a$, then there exists $c \in \mathcal{C}$ such that $cb \downarrow$ implies $cb = b$, and $bc \downarrow$ implies $bc = b$. The element c is unique, and is denoted 1_a or id_a , and called the *identity* on a . An *element* $f \in \mathcal{C}$ is an element f of the underlying set \mathcal{C} . An *object* a in \mathcal{C} is an element a of \mathcal{C} such that $\mathbf{s}(a) = \mathbf{t}(a) = a$. A *subject* U in \mathcal{C} is an element U of \mathcal{C} such that there exists $f \in \mathcal{C}$ such that $\mathbf{s}(f) = U$, or there exists $f \in \mathcal{C}$ such that $\mathbf{t}(f) = U$.

The approach of Definition 1 has the advantage of having fewer basic concepts than Definition 2. All elements are subjects and all elements have identities. This makes many steps of the development go smoothly. On the other hand, Definition 1 creates so many identities that one sometimes wonders if they are better avoided after all. Thus one might seem to be at an impasse concerning whether Definition 1 or Definition 2 is more preferable. This ambivalence is resolved by the notion of ideal category [14]. Ideal categories arise naturally in categorical logic. In such categories, and in particular in the generalized category of contexts \mathbf{CA} , there are identities present just as Definition 1 requires. This tipping of the scales is the reason why we favor Definition 1 over Definition 2. In order to facilitate discussions about generalized categories in the sense of Definition 1, we say that *closing over* 1_{\circ} is the obvious operation of ensuring (via free generation where needed) that axiom 5 is satisfied.

2.3. Resuming, from Definition 1.

Proposition 2.1. *Up to reversal of \sqsubseteq , Definition 1 is symmetric in the source and target maps \mathbf{s} and \mathbf{t} . Therefore every proof Φ about a generalized category \mathcal{C} continues to hold when, in all assumptions, definitions, and deduction steps, composition, the order \sqsubseteq , and the role of source and target are reversed.*

Such a proof Φ' is said to be obtained from Φ “by duality” [11]. This simple fact has a profound effect on the entire subject. The generalized category formed by the operation of Proposition 2.1 is called the *opposite generalized category* \mathcal{C}^{op} of \mathcal{C} .

Example 1. Let \mathcal{C} be a category [11]. Then *the generalized category generated by \mathcal{C}* is obtained from \mathcal{C} by identifying the identity 1_X of each object $X \in \mathcal{C}$ with X , and closing over 1_{\circ} . Considering a concrete example, such as the generalized category generated by the category of all groups, we may write id_X for X , with the identification $\text{id}_X = X$ being understood. More formally, we define:

Definition 3. A generalized category \mathcal{C} is a *category* or *one-category* if the source and target of every nonidentity f in \mathcal{C} is an object in \mathcal{C} .

We now have a rough ontology:

sharp category = category	proximal category
sharp generalized category	proximal generalized category = generalized category

Example 2. In some instances it is possible to write down a generalized category explicitly. There is an empty generalized category, and $\mathcal{C} = \{a : a \rightarrow a\}$, the trivial generalized category. More generally, any set S is a generalized category after setting $\mathbf{s}(a) = \mathbf{t}(a) = a$ for $a \in S$, we say that the generalized category is *discrete* or a *zero-category*, or simply that it is a set. (Thus, sets and categories are examples of generalized categories.) Because of the identity axiom, other than finite sets there are no finite generalized categories. To amend language, we therefore define:

Definition 4. A generalized category \mathcal{C} is *finitely generated* if there is a finite set \mathcal{C} such that the remainder of \mathcal{C} consists only of identities.

There are many examples of generalized categories that are not ordinary categories, the simplest perhaps being $\mathcal{C} = \{a : a \rightarrow a, b : a \rightarrow b\}$. Another simple example is $\mathcal{C} = \{a : b \rightarrow b, b : a \rightarrow a\}$. This generalized category is finite, but does not possess objects, moreover every element is a subject.

A generalized category may also lack objects due to infinite descent, for example $\mathcal{C} = \{a_n : a_{n-1} \rightarrow a_{n-1} \mid n \in \mathbb{Z}\}$.

Example 3. Besides the aforementioned sources in domain theory, motivation for the proximal relation \sqsubseteq in a proximal generalized category comes, via categorical semantics, from the subtyping relation in some type theoretical systems [12, 13], a feature characteristically found in object-oriented languages. Subtyped type systems are often preorders, thus, we can access the semantics given by a generalized category by, for one thing, equating mutual subtypes. As in domain theoretic orders, subtyped type systems often include a bottom type; they may also include a global maximum type. Such a structuring of types creates a comfortable intuitive environment for type theory, and makes the type checker behave less rigidly. However, representation of data in such systems can demand trade-offs that make such systems less suitable for some kinds of applications. Moreover, in an industry-level type system, problems and subtleties may arise due to the need for subtyping rules to interact coherently with rules that govern other advantageous type features, such as records, recursive types, and polymorphism. In practice, therefore, subtyping produces both benefits as well as costs, and has been the focus of much research and discussion in computer science. One approach to implementing subtyping involves data type *coercion*, or the automated physical modification of stored data at run-time. Type-theoretically, condition (4) of Definition 1 corresponds to a type system in which there exists a coercive evaluation mechanism.

Example 4. Let \mathcal{C} be a generalized category, and consider the condition on \mathcal{C} that hom sets should contain a unique element or else be empty. To obtain a (possibly infinite) planar binary tree one adds the condition that source and target may not loop except trivially, that is, for every element $a \in \mathcal{C}$, and for every finite sequence (x_1, \dots, x_n) where x_i is either **s** or **t** (source or target) if $x_n x_{n-1} \dots x_1 a = a$ then it is required that $\mathbf{s} a = \mathbf{t} a = a$, that is, or (using the terminology of trees) that a is a leaf. Presheaves on such trees arise for example in database theory, see for example [16].

Example 5. A *generalized (directed) graph* [14] is simply a triple $(\mathcal{A}, \mathbf{s}, \mathbf{t})$, where \mathcal{A} is a carrier set, and \mathbf{s}, \mathbf{t} are maps $\mathcal{A} \rightarrow \mathcal{A}$. An element of \mathcal{A} is (synonymously) an *edge*. An *object* in a generalized graph is an element $a \in \mathcal{A}$ such that $\mathbf{s} a = \mathbf{t} a = a$, that is, a common fixed point of the endomorphisms \mathbf{s} and \mathbf{t} . Ordinary graphs correspond bijectively with 1-dimensional generalized graphs, where we say that generalized graph is *1-dimensional* if $\mathbf{s} \mathbf{s} = \mathbf{s}$ and $\mathbf{t} \mathbf{t} = \mathbf{t}$. With the obvious composition via compound paths, a generalized graph becomes a (sharp) generalized category.

There are plentiful settings where generalized graphs may arise. For example, suppose that there is a system of goods \mathcal{A}_0 . The edges of \mathcal{A} are certificates (issued, say perhaps, by different governing bodies) that say that a good $a \in \mathcal{A}_0$ may be exchanged for another good $b \in \mathcal{A}_0$. Suppose it is accepted that a good is always exchangeable for itself. Now let's suppose that such certificates themselves may be exchanged, but that this requires that one has a higher-level certificate for this higher-level trade. If we imagine a certain impetus exists among those we imagine making the exchanges, we can expect that there will next arise trading for these certificates as well, giving rise to a generalized graph (in fact, a generalized deductive system, via a simple extension of Kolmogorov's reasoning about intuitionistic logic in [9]).

Example 6. For a planar binary tree \mathbf{t} , let

root(\mathbf{t}) is the root of \mathbf{t} .

left(\mathbf{t}) is the tree given by the left descendant of the root, and its descendants.

right(\mathbf{t}) is the tree given by the right descendant of the root, and its descendants.

From any category \mathcal{C} we can form a sharp generalized category $\mathcal{C}f$ as follows: take the set $\mathcal{C}f$ to be the set of all planar binary trees of morphisms in \mathcal{C} , subject to a source-and-target condition

$$\text{dom root}(\text{dom } f) = \text{dom root}(f),$$

and

$$\text{cod root}(\text{cod } f) = \text{cod root}(f),$$

where if f be such a tree,

$$\text{cod } f = \text{left}(f),$$

the left descendent tree of f , and

$$\text{dom } f = \text{right}(f),$$

the right descendent tree of f . These conditions set up a recursive condition on elements of $\mathcal{C}f$.

For $g, f \in \mathcal{C}f$, we set

$$g \cdot f := (\text{ the tree } h \text{ with left descendent } \text{root}(f), \text{ right descendent } \text{root}(g) \text{ and } \text{root } \text{root}(g) \cdot \text{root}(f).).$$

This is a well-defined product, by the source-and-target condition above. It is checked that this is a (sharp) generalized category. An element of $\mathcal{C}f$ may be visualized as

$$\begin{array}{ccc} & & \bullet \\ & & \vdots \hat{f} \\ & & \downarrow \\ X & \xrightarrow{\quad f \quad} & Y \\ & & \vdots \bar{f} \\ & & \downarrow \\ & & \bullet \end{array}$$

Constructions on the original \mathcal{C} can be carried over to $\mathcal{C}f$, for example, if \mathcal{C} has products (equalizers, coproducts, coequalizers), then so (respectively) does $\mathcal{C}f$. If \mathcal{C} is (co)complete, however, it does not imply that $\mathcal{C}f$ is (co)complete, see [14].

Lawvere's comma category construction [10, 11] may also be observed to yield generalized categories, even when the input data is an ordinary category. Fix two generalized categories \mathcal{C} , \mathcal{D} , and \mathcal{E} , and functors $S : \mathcal{D} \rightarrow \mathcal{C}$ and $T : \mathcal{E} \rightarrow \mathcal{C}$. Let

$$(S, T) := \{(d, e, f) \mid d \in \mathcal{D}, e \in \mathcal{E}, f \text{ is a planar binary tree of pairs } (f, g), f, g \in \mathcal{C}\}$$

Set

$$\begin{aligned} \overline{(d, e, f)} &= (\bar{d}, \bar{e}, \text{right}(f)), \\ \widehat{(d, e, f)} &= (\hat{d}, \hat{e}, \text{left}(f)). \end{aligned}$$

Composition in (S, T) is defined as in $\mathcal{C}f$.

3. ELEMENTARY THEORY, CATEGORY OF INVERTIBLES

We now define functors and hom sets:

Definition 5. A mapping $\mathcal{C} \rightarrow \mathcal{C}'$ between generalized categories is *functorial* or a *functor* if

- (1) $a \sqsubseteq b$ implies $F(a) \sqsubseteq F(b)$,
- (2) $F(\bar{a}) = \overline{F(a)}$,
- (3) $F(\hat{a}) = \widehat{F(a)}$,
- (4) $F(ab) = F(a)F(b)$, if $ab \downarrow$,
- (5) $F(1_a) = 1_{F(a)}$.

We thus have a category **GenCat** of generalized categories and functors.

Functors are also called *covariant functors*. A *contravariant functor* from \mathcal{C} to \mathcal{C}' is a unital map satisfying

- (1) if $a \sqsubseteq b$ then $F(b) \sqsubseteq F(a)$,
- (2) $F(\bar{a}) = \widehat{F(a)}$,
- (3) $F(\hat{a}) = \overline{F(a)}$,
- (4) $F(ab) = F(b)F(a)$ if $ab \downarrow$,

instead of the corresponding covariant relations.

Definition 6. The sets

$$\text{hom}(a, b) = \{c \in \mathcal{C} \mid \bar{c} = a, \hat{c} = b\},$$

for $a, b \in \mathcal{C}$, are called the *hom sets* of \mathcal{C} .

Definition 7. A *subcategory* of a generalized category \mathcal{C} is a subset \mathcal{C}' of \mathcal{C} whose order is inherited from \mathcal{C} closed under source, target, composition, and identities: if $a \in \mathcal{C}'$, then $1_a \in \mathcal{C}'$. A subcategory \mathcal{C}' is *full* if $a, b \in \mathcal{C}'$ implies $\text{hom}(a, b)$ is contained in \mathcal{C}' .

The composition of two functors is a functor, and functors send objects to objects.

Definition 8. Two generalized categories \mathcal{C} and \mathcal{C}' are *isomorphic* if there is an invertible functor (i.e., invertible as a mapping) F from \mathcal{C} to \mathcal{C}' .

Proposition 3.1. *There is a functor, flattening, from the category of generalized categories to the category of categories.*

Proof. Let \mathcal{C} be a generalized category with identities. Let $\text{Ob}(\mathcal{C}_{\text{flat}})$ be $\{[f] \mid f \in \mathcal{C}\}$, the objects of \mathcal{C} indexed by the elements of \mathcal{C} . Let $\text{Mor}(\mathcal{C}_{\text{flat}})$ again be a set $\{(f) \mid f \in \mathcal{C}\}$ indexed by the elements of \mathcal{C} , and define source and target

$$\begin{aligned} s((f)) &= [s(f)], \\ t((f)) &= [t(f)]. \end{aligned}$$

Then $\mathcal{C}_{\text{flat}}$ is a category whose composition and identities are

$$\begin{aligned} (g) \cdot (f) &:= (gf), \\ 1_{[f]} &= (1_f). \end{aligned}$$

Given a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ in **GenCat**, we immediately obtain a functor $\mathcal{C}_{\text{flat}} \rightarrow \mathcal{D}_{\text{flat}}$. \square

Note that $\mathcal{C}_{\text{flat}}$ contains a flattening of the identity structure, even in cases where $\text{hom}(a, a) = \{1_a\}$.

There is also a category $\text{flat}\mathcal{C}$, the further flattening of \mathcal{C} to a zero-category. It is defined by:

$$\text{flat}(f) := \begin{cases} (f), & \text{if } f = 1_g \text{ for some } g \in \mathcal{C}, \\ [f] & \text{otherwise,} \end{cases}$$

where $[f]$ is defined by $\mathbf{s}([f]) = \mathbf{t}([f]) = [f]$, and $(f) : \text{flat}(\mathbf{s}(f)) \rightarrow \text{flat}(\mathbf{t}(f))$.

Definition 9. If \mathcal{C} is a generalized category, an element $a \in \mathcal{C}$ is *invertible* if there exists $b \in \mathcal{C}$ such that $ab = 1_{\hat{a}}$ and $ba = 1_{\bar{a}}$.

Proposition 3.2.

- (1) *The inverse a^{-1} of an element a of \mathcal{C} is unique if it exists.*
- (2) *$\widehat{a^{-1}} = \bar{a}$ and $\overline{a^{-1}} = \hat{a}$. (Even if \mathcal{C} is proximal.)*

- (3) All objects a are invertible: $a^{-1} = a$.
- (4) Functors send invertibles to invertibles: $F(\theta^{-1}) = F(\theta)^{-1}$.

There are a few ways a generalized category may be partitioned into equivalence classes:

Definition 10. For $a, b \in \mathcal{C}$, we have the following equivalence relations:

- (1) a and b are in the same *monic class*, or *subobject*, $a \sim_m b$, if there exists invertible element $\theta \in \mathcal{C}$ such that $a\theta = b$.
- (2) a and b are in the same *epic class*, or *quotient*, $a \sim_e b$, if there exists invertible element $\theta \in \mathcal{C}$ such that $\theta a = b$;
- (3) a and b are in the same *iso class*, $a \sim b$, if there exist invertible elements $\theta_1, \theta_2 \in \mathcal{C}$ such that $\theta_1 a = b\theta_2$.

Let Θ denote the set of all invertible elements in \mathcal{C} . Define the symbol

$$a\Theta := \{a \cdot \theta \mid \theta \in \Theta \text{ and } a \cdot \theta \downarrow\},$$

and define the symbols $\Theta a, \Theta a\Theta$, etc. similarly. Then for $a, b \in \mathcal{C}$, b belongs to the monic class of a if and only if $b \in a\Theta$, b belongs to the epic class of a if and only if $b \in \Theta a$, and b belongs to the iso class of a if and only if $b \in \Theta a\Theta$. This notation is useful for back-of-the-envelope calculations, but it can be misleading: it need not be true that $\Theta f\Theta = \Theta g\Theta$, even if f and g are invertible.

Definition 11. An element m of a generalized category \mathcal{C} is *monic* if $mf, mg \downarrow$ and $mf = mg$ implies $f = g$. An element e in \mathcal{C} is *epi* if $fe, ge \downarrow$ and $fe = ge$ implies $f = g$. We say a is *isomorphic* to b , denoted

$$a \cong b,$$

if there exists an invertible element θ with $\bar{\theta} = a, \hat{\theta} = b$.

If a is monic and $a \sim_m b$, then b is monic, and the θ given by the definition is unique. Similarly, if a is epic and $a \sim_e b$.

For every $a, b \in \mathcal{C}$, a is isomorphic to b iff 1_a is in the same iso class as 1_b , that is,

$$a \cong b \iff 1_a \sim 1_b.$$

For a, b objects, this becomes:

$$a \cong b \iff a \sim b.$$

Proposition 3.3. *Let \mathcal{C} be a generalized category. Then the set of iso classes forms a sharp category. The objects of this category are the iso classes of invertible elements of \mathcal{C} .*

Proof. Let $\tilde{\mathcal{C}}$ be the set of iso classes of \mathcal{C} , let $\tilde{a}, \tilde{b}, \dots$ denote elements in $\tilde{\mathcal{C}}$. Define

$$\tilde{a} \cdot \tilde{b} := \{\theta_1 a \theta_2 b \theta_3 \mid \theta_1, \theta_2, \theta_3 \text{ invertible, and } \theta_1 a \theta_2 b \theta_3 \downarrow\}.$$

This is a partially defined map $\tilde{\mathcal{C}} \times \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}}$. For $a \in \mathcal{C}$, let

$$\tilde{\mathbf{s}}(\tilde{a}) := \widetilde{1_{\mathbf{s}a}},$$

$$\tilde{\mathbf{t}}(\tilde{a}) := \widetilde{1_{\mathbf{t}a}}.$$

These operations are well-defined: if $a = \theta_1 b \theta_2$, then \tilde{a} is isomorphic to \tilde{b} , so, say, $\theta_1 b \theta^{-1} = 1_{\tilde{a}}$, so $\widetilde{1_{\mathbf{s}a}} = \widetilde{1_{\mathbf{s}b}}$, and similarly for $\tilde{\mathbf{t}}$.

We take the order \sqsubseteq on $\tilde{\mathcal{C}}$ to be trivial, and we check Definition 1. The first four conditions are immediate: for (4), if $\tilde{a}, \tilde{b} \in \tilde{\mathcal{C}}$, then $\tilde{a}\tilde{b} \downarrow$. This occurs if and only if $\{\theta \in \mathcal{C} \mid \theta : \tilde{a} \rightarrow \tilde{b} \text{ is invertible}\}$

is nonempty, if and only if $1_{\tilde{a}} \sim 1_{\tilde{b}}$, if and only if $\tilde{s}(\tilde{a}) = \tilde{t}(\tilde{b})$. Next, we observe that if \tilde{a} is an element of the form $\tilde{s}\tilde{b}$ or $\tilde{t}\tilde{b}$ in $\tilde{\mathcal{C}}$, then it must be of the form $\tilde{1}_b$ for some $b \in \mathcal{C}$, and

$$\tilde{t}(\tilde{1}_b) = \tilde{s}(\tilde{1}_b) = \tilde{1}_b,$$

so $\tilde{1}_b$ is an object. Next, we have

$$\tilde{1}_a \cdot \tilde{b} = \{\theta_1 1_a \theta_2 b \theta_3\} = \{\theta_4 b \theta_3\} = \tilde{b},$$

and similarly, $\tilde{b}\tilde{1}_a = \tilde{b}$ whenever the product is defined. So $\tilde{\mathcal{C}}$ is a sharp generalized category, in fact a one-category, after closing over $1_{\langle \rangle}$. The second statement is merely the observation that a is invertible if and only if $\tilde{a} = \widetilde{1_{\mathbf{s}a}} = \widetilde{1_{\mathbf{t}a}}$. \square

Definition 12. We refer to the category $\tilde{\mathcal{C}}$ of Proposition 3.3 as the *category of invertibles* of \mathcal{C} .

The *skeleton* of a generalized category \mathcal{C} is any full subcategory such that each element of \mathcal{C} is isomorphic in \mathcal{C} to exactly one element of the subcategory. Skeletons are unique up to isomorphism [11]. In the case of a category \mathcal{C} , the category of invertibles expresses exactly the same data as a skeleton, but in a different way: any iso class that is an object in the category of invertibles contains not a set of invertibles in \mathcal{C} that are pairwise isomorphic, but instead, the set of all the isomorphisms that relate them pairwise to one another. On the other hand, an iso class that is an arrow in the category of invertibles is a noninvertible arrow $f \in \mathcal{C}$ well-defined up to a commutative square with invertible columns.

Since every element has an identity, thus taking the category of invertibles is the same as the operation of flattening (Proposition 3.1) followed by taking the skeleton, yielding the description just made in the previous paragraph. Thus it is perhaps natural to think of it as the “category of identities” of the generalized category.

It is also the case that a functor F lifts to a functorial map \tilde{F} on the category of invertibles. Indeed, define

$$\tilde{F} : \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}}',$$

via

$$\tilde{F}(\tilde{a}) := \widetilde{F(a)}.$$

This is well-defined, as a consequence of (2) (which depends on the unital property of F):

$$\tilde{F}(\theta_1 a \theta_2) = \widetilde{F(\theta_1) F(a) F(\theta_2)} = \widetilde{F(a)}.$$

So we check functoriality: we have

$$\widetilde{1_{\mathbf{s}(F(a))}} = \widetilde{1_{F(\mathbf{s}(a))}} = \widetilde{F(1_{\mathbf{s}(a)})} = \tilde{F}(\widetilde{1_{\mathbf{s}(a)}}) = \tilde{F}(\mathbf{s}(\tilde{a})),$$

and

$$\widetilde{1_{\mathbf{s}(F(a))}} = \mathbf{s}(\widetilde{F(a)}) = \mathbf{s}(\tilde{F}(\tilde{a})).$$

Similarly,

$$\mathbf{t}(\tilde{F}(\tilde{a})) = \tilde{F}(\mathbf{t}(\tilde{a})).$$

And

$$\tilde{F}(\tilde{a}\tilde{b}) = \tilde{F}(\widetilde{a\theta b}) = \widetilde{F(a)\theta' F(b)} = \widetilde{F(a)}\theta' \tilde{F}(\tilde{b}) = \tilde{F}(\tilde{a})\tilde{F}(\tilde{b}).$$

Finally, \tilde{F} is unital since $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}'$ are categories.

A notion weaker than isomorphism arises from considering the categories of invertibles.

Definition 13. Generalized categories \mathcal{C} and \mathcal{C}' are *equivalent* if their categories of invertibles are isomorphic.

This definition appeals directly to a comparison of the categories of invertibles. Now consider two functors $F, G : \mathcal{C} \rightarrow \mathcal{C}'$ that both define the same functor $\tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}}'$ on the categories of invertibles of \mathcal{C} and \mathcal{C}' . This can only mean that there exist a pair of functions $\theta_1, \theta_2 : \mathcal{C} \rightarrow \mathcal{C}'$ such that $\forall a \in \mathcal{C}$ $\theta_i(a)$ is invertible for $i = 1, 2$, and for all $a \in \mathcal{C}$,

$$\theta_1(a)F(a) = G(a)\theta_2(a) \downarrow.$$

If this holds we may write

$$F \cong G.$$

Proposition 3.4. *Two generalized categories \mathcal{C} and \mathcal{C}' are equivalent if either of the following two equivalent conditions are satisfied.*

- (1) *Their categories of invertibles are isomorphic via a pair \tilde{F}, \tilde{G} , where $\tilde{G} = \tilde{F}^{-1}$, that come from functors $F : \mathcal{C} \rightarrow \mathcal{C}'$ and $G : \mathcal{C}' \rightarrow \mathcal{C}$.*
- (2) *There exist two functors F, G from $\mathcal{C} \rightarrow \mathcal{C}'$ ($\mathcal{C}' \rightarrow \mathcal{C}$, respectively) satisfying*

$$F \circ G \cong \text{id}_{\mathcal{C}'},$$

$$G \circ F \cong \text{id}_{\mathcal{C}}.$$

We can consider properties that a functor \tilde{F} on the category of invertibles has as an ordinary functor, and view them as properties of the underlying functor F :

Definition 14. A functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ is *essentially injective* if it satisfies one of the following equivalent conditions,

- (1) \tilde{F} is injective.
- (2) For $a, b \in \mathcal{C}$, $F(a) = F(b)$ implies $a \sim b$.

and F is *essentially surjective* if it satisfies one of the following equivalent conditions:

- (1) \tilde{F} is surjective.
- (2) For $\alpha \in \mathcal{C}'$, there exists $a \in \mathcal{C}$ with $F(a) \sim \alpha$.

From our initial investigation of equivalences between generalized categories, we arrived at the notion of equivalence via a pair of functors F and G . We could, however, view this machinery (the pair (θ_1, θ_2)) as instead relating the two functors, and extend it:

Definition 15. Let $\mathcal{C}, \mathcal{C}'$ be generalized categories, let $F, G : \mathcal{C} \rightarrow \mathcal{C}'$ be two functors. We say that a *morphism of functors* [8] from F to G is a pair (θ_1, θ_2) of maps $\mathcal{C} \rightarrow \mathcal{C}'$ satisfying, for all $a \in \mathcal{C}$,

$$(1) \quad \theta_1(a)F(a) = G(a)\theta_2(a) \downarrow$$

Note that here, θ_1 and θ_2 are no longer presumed to be invertible. We may write the morphism of functors with the notation $(\theta_1, \theta_2) : F \Rightarrow G$.

Note that the maps θ_1 and θ_2 are maps from \mathcal{C} to \mathcal{C}' , not from $\text{Ob}(\mathcal{C})$ to \mathcal{C}' (cf. [11]).

Example 7. Let $A = (a_{ij})$ be a matrix with coefficients in a ring R , and let $f : R \rightarrow S$ be a ring homomorphism. One naturally sets $f(A) = (f(a_{ij}))$, and doing this, one sees that

$$(2) \quad \det(f(A)) = f(\det(A)).$$

This relation can be interpreted by observing that GL_n is a functor from the category of rings to the category of groups, and likewise for the mapping that sends a ring to its group of units, and a ring homomorphism to the pointwise-identical homomorphism on the respective groups of units.

So if $f : R \rightarrow S$, and writing $F(f)$ for the map defined above extending f to a map on $GL_n(R)$, and $G(f)$ for the map changing f to a map on the group of units, we have

$$\det() \circ F(f) = G(f) \circ \det()$$

by rewriting equation (2). From this expression we can read off the morphism of functors:

$$\theta_1(f) = \det : GL_n(S) \rightarrow S^\times,$$

$$\theta_2(f) = \det : GL_n(R) \rightarrow R^\times.$$

We see that in this example, θ_1 and θ_2 come from a single map θ on the objects (rings). This is not only typical of categories, it is guaranteed to happen. Indeed, if we return to the general situation of Definition 15, inserting $a = 1_b$ into equation (1) gives

$$\theta_1(1_b) = \theta_2(1_b)$$

for $b \in \mathcal{C}$, so in particular, for all objects b ,

$$\theta_1(b) = \theta_2(b).$$

Thus θ_1 and θ_2 are identical on objects, and since one-categories have no higher morphisms, this single map on objects completely characterizes (θ_1, θ_2) .

In the terminology of section 4 that follows, this means that a morphism of functors between functors relating categories is always *natural*. In the setting of generalized categories, we might suppose that this naturality property is a condition special to one-categories, since it does not appear to have any a priori motivation. However, the theory that results from dropping the naturality condition appears to be significantly weaker:

- (1) There is no strict 2-category of non-natural transformations, functors, and generalized categories. Here, the wheel turns on the tiniest of pedestals: in the notation of Appendix A, the relations

$$\begin{aligned}\bar{\alpha}(X) &= \overline{\alpha(X)}, \\ \hat{\alpha}(X) &= \widehat{\alpha(X)}\end{aligned}$$

hold only in the natural setting. So we do not prove Fact 1.

- (2) While there is a notion of non-natural adjunction, there is no hom set bijection. A key step in the proof uses the naturality of the unit and counit maps. This in turn is used to prove that left adjoints are right exact.
- (3) Because there is no adjoint hom set bijection, some theorems relating equivalences of categories with properties of functors no longer hold. In particular a full, faithful, essentially surjective functor might not define an equivalence.

For these reasons, we do not take the development any further until we introduce naturality in the next section.

4. NATURALITY

In this section we establish the second of the two notions of equivalence we consider, namely natural equivalence. As already noted, *the distinction between natural and non-natural vanishes in the case of categories*. Under natural equivalence, we obtain a 2-category of generalized categories, and in particular, an interchange law (Theorem 4.1). We can also establish, using the final lynchpin that naturality provides, the hom set bijection associated with adjoint pairs (Theorem 4.2). Consequently the familiar rule that an equivalence between categories is given by a fully

faithful essentially surjective functor carries over to generalized categories (Theorem 4.3). The full and faithful properties are tied to the naturality condition, which gives rise to maps not only on individual elements, but on entire hom sets.

Definition 16. Let \mathcal{C}, \mathcal{D} be generalized categories, let $F, G : \mathcal{C} \rightarrow \mathcal{D}$. Let $(\theta_1, \theta_2) : F \Rightarrow G$ be a morphism of functors. We say that (θ_1, θ_2) is *natural* or that (θ_1, θ_2) is a *natural transformation* if, for every $a, b \in \mathcal{C}$,

$$\theta_1(a) = \theta_1(b)$$

whenever $\hat{a} = \hat{b}$, and

$$\theta_2(a) = \theta_2(b)$$

whenever $\bar{a} = \bar{b}$.

Thus, naturality means that the function $\theta_1(a)$ can be replaced with the function $\hat{a} \mapsto \theta_1(1_{\hat{a}})$ of the element \hat{a} , and θ_2 can be replaced with the function $\bar{a} \mapsto \theta_2(1_{\bar{a}})$ of the element \bar{a} . But, as noted in section 3, $\theta_1(1_b) = \theta_2(1_b)$ for all elements b . Hence a natural transformation reduces to a single map $\theta : \mathcal{C} \rightarrow \mathcal{C}'$, from which θ_1 and θ_2 are immediately derived:

$$\theta_1(a) := \theta(1_{\hat{a}}),$$

$$\theta_2(a) := \theta(1_{\bar{a}}).$$

We refer to a natural transformation (θ_1, θ_2) by referring to this map θ . In terms of θ the defining relation of a morphism of functors becomes

$$\theta(\hat{f}) \cdot F(f) = G(f) \cdot \theta(\bar{f}) \downarrow.$$

Definition 17. Two generalized categories \mathcal{C} and \mathcal{C}' are *naturally equivalent* if they are equivalent via natural transformations

$$\theta : F \circ G \cong \text{id}_{\mathcal{C}'},$$

$$\theta' : G \circ F \cong \text{id}_{\mathcal{C}}.$$

Naturally equivalent generalized categories are, in particular, equivalent (Definition 13). With the extra condition of naturality, the way is clear to extend many justly well-known results of one-category theory [11] to the generalized setting:

Theorem 4.1. *The system given by all of the generalized categories, functors, and natural transformations forms a strict 2-category.*

Proof. We define the products

$$\theta_1 \triangle \theta_2,$$

$$\theta_1 \star \theta_2$$

just as in Appendix A, and proceed as in the one-categorical case. □

We include the naturality condition when defining adjoints:

Definition 18. Let \mathcal{C} and \mathcal{D} be generalized categories. An *adjunction* $(F, G, \eta, \varepsilon)$ is a pair of functors

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D}$$

together with natural transformations

$$\eta : \text{id}_{\mathcal{C}} \rightarrow G \circ F, \quad \varepsilon : F \circ G \rightarrow \text{id}_{\mathcal{D}},$$

satisfying the identities

$$(3) \quad (G \circ \varepsilon) \triangle (\eta \circ G) = 1_G,$$

$$(4) \quad (\varepsilon \circ F) \triangle (F \circ \eta) = 1_F,$$

where 1_F is the mapping $f \mapsto 1_{F(f)}$. Given an adjunction $(F, G, \eta, \varepsilon)$, η is called the *unit* and ε is called the *counit* of the adjunction. A natural equivalence (θ, θ') is an *adjoint equivalence* if θ and θ' are the unit and counit of an adjunction.

Theorem 4.2. *Let \mathcal{C}, \mathcal{D} be generalized categories, and let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be functors. The following are equivalent:*

- (1) $(F, G, \eta, \varepsilon)$ forms an adjunction $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$.
- (2) For every f in \mathcal{C} and g in \mathcal{D} , there is a bijection of sets
- (5) $\text{hom}(F(f), g) \cong \text{hom}(f, G(g))$,

that is natural in f and g . This means that if $\phi_{f,g}$ is the bijection (5), then for every $k : g \rightarrow g'$, and $h : f' \rightarrow f$, the following diagrams commute:

$$\begin{array}{ccc} \text{hom}(F(f), g) & \xrightarrow{\phi_{f,g}} & \text{hom}(f, G(g)) \\ k_* \downarrow & & \downarrow G(k)_* \\ \text{hom}(F(f), g') & \xrightarrow{\phi_{f,g'}} & \text{hom}(f, G(g')) \end{array} \quad \begin{array}{ccc} \text{hom}(F(f), g) & \xrightarrow{\phi_{f,g}} & \text{hom}(f, G(g)) \\ F(h)^* \downarrow & & \downarrow h^* \\ \text{hom}(F(f'), g) & \xrightarrow{\phi_{f',g}} & \text{hom}(f', G(g)) \end{array}$$

Equivalently ϕ satisfies

$$u \cdot F(v) : F(f) \rightarrow g \text{ implies } \phi(u \cdot F(v)) = \phi(u) \cdot v,$$

$$v' \cdot v : F(f) \rightarrow g \text{ implies } \phi(v' \cdot v) = G(v') \cdot \phi(v).$$

Proof. The proof is formally the same as in the one-categorical case (see [11]). \square

Definition 19. Let \mathcal{C}, \mathcal{D} be generalized categories, $F : \mathcal{C} \rightarrow \mathcal{D}$ a functor. For $a, b \in \mathcal{C}$, let $F_{a,b}$ be the mapping on the domain $\text{hom}(a, b)$ given by $f \mapsto F(f)$. We say that F is *faithful* if for all a, b , $F_{a,b}$ is injective, and we say that F is *full* if for all a, b , $F_{a,b}$ is surjective.

Thus for example full means: if α, β in \mathcal{D} are of the form $F(a), F(b)$, for $a, b \in \mathcal{C}$, and if $\gamma : \alpha \rightarrow \beta$, then γ is of the form $F(c)$ for $c \in \mathcal{C}$.

Theorem 4.3. *Let \mathcal{C}, \mathcal{D} be generalized categories, and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. The following are equivalent:*

- (1) F is a natural equivalence,
- (2) F is a natural adjoint equivalence,
- (3) F is full, faithful, and essentially surjective.

Proof. The proof, much the same as in the one-categorical case, is left to the reader. \square

5. LIMITS

In this section we establish the elements of the theory of limits and colimits in sharp generalized categories. We consider limits with respect to mappings $I \rightarrow \mathcal{C}$ as in Definition 22 that are weaker than functors. This, for example, allows us to form the shape of a product or coproduct of any set of elements in a generalized category.

Definition 20. Let $\mathcal{C}, \mathcal{C}'$ be generalized categories. A *functor up to objects* from \mathcal{C} to \mathcal{C}' is a map $F : \mathcal{C} \rightarrow \mathcal{C}'$ satisfying, for every $a, b \in \mathcal{C}$,

- (1) $F(ab) = F(a)(b)$,
- (2) $F(a)$ is an identity in \mathcal{C}' if and only if a is an identity in \mathcal{C} ,
- (3) $F(\mathbf{s}(a)) = \mathbf{s}(F(a))$ unless a is an object of \mathcal{C} ,
- (4) $F(\mathbf{t}(a)) = \mathbf{t}(F(a))$ unless a is an object of \mathcal{C} .

Definition 21. Let \mathcal{C} be a generalized category, I a generalized category (the index of a cone needs only be a set, but in practice it is always a (generalized) category). A *cone* in \mathcal{C} with index I is a map $\sigma : I \rightarrow \mathcal{C}$ such that

$$\text{for all } i, j \in I, \overline{\sigma(i)} = \overline{\sigma(j)}.$$

Dually, *cocone* in \mathcal{C} with index I is a map $\sigma : I \rightarrow \mathcal{C}$ such that for all $i, j \in I$, $\widehat{\sigma(i)} = \widehat{\sigma(j)}$. A cone or cocone is *finitely generated* if the index set I is finitely generated (Definition 4). This common source is the *vertex* of the cone, and the *vertex* of a cocone is the common target. Given a cone or cocone π , we may refer to $\pi(i)$ for some $i \in I$ as a *member* of the cone.

Definition 22. Let \mathcal{C}, I be generalized categories. Let $\alpha : I \rightarrow \mathcal{C}$ be a functor, possibly only a functor up to objects. A cone is said to be *over (or below) the base α* if

- (1) $\widehat{\pi(i)} = \alpha(i)$, for all $i \in I$,
- (2) for all $i \in I$, $\pi(\hat{i}) = \alpha(i)\pi(\bar{i})$.

A *limit* of α is a cone $\pi : I \rightarrow \mathcal{C}$ below the base α such that for any cone $\tilde{\pi} : I \rightarrow \mathcal{C}$ over the same base α , there is a unique $\lambda \in \mathcal{C}$ such that $\tilde{\pi} = \pi \triangle \lambda$. (Here, $\pi \triangle \lambda$ is the map defined by $(\pi \triangle \lambda)(i) = \pi(i) \cdot \lambda$.)

Dually, a cocone is said to be *over (or below) the base α* if

- (1) $\overline{\pi(i)} = \alpha(i)$, for all $i \in I$,
- (2) for all $i \in I$, $\pi(\bar{i}) = \pi(\hat{i})\alpha(i)$.

A *colimit* of α is a cocone $\pi : I \rightarrow \mathcal{C}$ such that for any cone $\tilde{\pi} : I \rightarrow \mathcal{C}$ over the base α , there is a unique $\lambda \in \mathcal{C}$ such that $\tilde{\pi} = \lambda \triangle \pi$. Here, $\lambda \triangle \pi$ is the map defined by $(\lambda \triangle \pi)(i) = \lambda \cdot \pi(i)$, as before.

Thus a cone fits a pattern as in the following Figure:

$$\begin{array}{ccc} \alpha(\bar{i}) & \xrightarrow{\alpha(i)} & \alpha(\hat{i}) \\ \swarrow \pi(\bar{i}) & & \nearrow \pi(\hat{i}) \\ & \text{(vertex)} & \end{array}$$

The word limit is often used to refer to the domain of the cone, and similarly colimit is used to refer to the codomain of the cocone. The terms *product*, *equalizer*, *coproduct*, *coequalizer*, etc. retain their meaning from ordinary categories, referring to limits based on diagrams $\alpha : I \rightarrow \mathcal{C}$ of the same shape as in the one-categorical case, and where α may be a functor only up to objects. We

follow standard terminology and say that a generalized category *has finite limits* if there is a limit cone for every finitely generated diagram $\alpha : I \rightarrow \mathcal{C}$, and dually for colimits.

We denote the set of limits of the functor $\alpha : I \rightarrow \mathcal{C}$ by $\lim(\alpha, I)$ or just $\lim \alpha$. We denote the colimit $\text{colim}(\alpha, I)$ or simply $\text{colim}(\alpha)$.

If \mathcal{C} is a generalized category, there exist (finitely generated) diagrams $J \rightarrow \mathcal{C}$ that cannot be defined and do not exist in an ordinary category. However, we still have:

Theorem 5.1. *Let \mathcal{C} be a generalized category. For \mathcal{C} to have all finite limits, it suffices that \mathcal{C} has all finite products and equalizers.*

Proof. We proceed by induction on the height of finitely generated diagrams $\alpha : I \rightarrow \mathcal{C}$. A finitely generated diagram of height 0 is a finite product, hence it has a limit cone in \mathcal{C} by hypothesis. Suppose that all finitely generated diagrams of height $k \geq 0$ have a limit cone, and let $\alpha : I \rightarrow \mathcal{C}$ be a diagram of height $k + 1$. Define

$$\alpha^{\leq k}$$

to be α restricted to the generalized category $I^{\leq k}$ formed by taking the collection of all elements of I of height $\leq k$, along with all identities of I . It is easy to see that $I^{\leq k}$ is closed under composition, thus it is a generalized category. Therefore $\alpha^{\leq k}$ is a diagram on \mathcal{C} , and by hypothesis, has a limit cone $\sigma^{\leq k}$ with vertex, say, $L^{\leq k}$. Consider $\text{flat}(I^{\leq k})$, the flattening of $I^{\leq k}$ to a zero-category (section 3). The diagram $\text{flat}(\alpha^{\leq k}) : \text{flat}(I^{\leq k}) \rightarrow \mathcal{C}$ induced by $\alpha^{\leq k}$ is a diagram of height zero, so it has a limit cone $\sigma^{\leq k, \text{flat}}$, with vertex, say, $L^{\leq k, \text{flat}}$. The cone $\sigma^{\leq k}$ on $I^{\leq k}$ induces a cone on $\text{flat}(I^{\leq k})$, so there exists a universal arrow

$$u_1 : L^{\leq k} \rightarrow L^{\leq k, \text{flat}}.$$

Now let $I^{k+1, \text{flat}}$ be the flattened (to a zero category) elements of I of height $k + 1$. The diagram α induces a diagram $\alpha^{k+1, \text{flat}}$ on $I^{k+1, \text{flat}}$, defined by

$$\alpha^{k+1, \text{flat}}(i) := \mathbf{t}(\alpha(i)).$$

This diagram (of height zero) has a limit cone $\sigma^{k+1, \text{flat}}$ with vertex, say, $L^{k+1, \text{flat}}$. For $i \in I$ of height $k + 1$, let π_i be the element in \mathcal{C} which is the projection

$$\pi_i : L^{\leq k, \text{flat}} \rightarrow \mathbf{t}(\alpha(i)),$$

coming from the diagram $\sigma^{\leq k, \text{flat}}$ on $I^{\leq k, \text{flat}}$ (where our notation hides this fact about π_i).

The previous cone $\sigma^{\leq k, \text{flat}}$ with vertex $L^{\leq k, \text{flat}}$ itself has projection arrows to the elements $\mathbf{t}(\alpha(i))$ as i ranges over $\alpha^{k+1, \text{flat}}$. Therefore, there is a universal arrow

$$u_2 : L^{\leq k, \text{flat}} \rightarrow L^{k+1, \text{flat}}.$$

Moreover, for each i of height $k + 1$, there is also a projection arrow to the element $\mathbf{s}(\alpha(i))$, and composing each of these projection arrows with $\alpha(i)$ gives a second cone with the same vertex $L^{\leq k, \text{flat}}$ on the diagram $\alpha^{k+1, \text{flat}}$. So we may again find a universal arrow

$$u_3 : L^{\leq k, \text{flat}} \rightarrow L^{k+1, \text{flat}},$$

by applying the universal property of the limit with vertex $L^{k+1, \text{flat}}$ a second time. We compose u_2 and u_3 with u_1 to form parallel arrows, and take the equalizer:

$$L \xrightarrow{e} L^{\leq k} \xrightarrow{u_1} L^{\leq k, \text{flat}} \xrightarrow[u_3]{u_2} L^{k+1, \text{flat}}$$

Now we define, for i in I of height $\leq k + 1$,

$$\sigma^{\leq k+1}(i) := \pi_i \cdot u_1 \cdot e.$$

We claim that this is a limit cone for the diagram $\alpha^{\leq k+1} : I^{\leq k+1} \rightarrow \mathcal{C}$. Since we pass through e to reach $L^{\leq k+1}$, $\sigma^{\leq k+1}$ satisfies $\sigma^{\leq k+1}(\hat{i}) = \alpha^{\leq k+1}(i) \cdot \sigma^{\leq k+1}(\hat{i})$, hence is a limit cone. Suppose that $\tilde{\sigma}^{\leq k+1} : I^{\leq k+1} \rightarrow \mathcal{C}$ is a diagram with vertex, say, \tilde{L} satisfying $\tilde{\sigma}^{\leq k+1}(\hat{i}) = \alpha^{\leq k+1}(i) \tilde{\sigma}^{\leq k+1}(\bar{i})$. Then $\tilde{\sigma}^{\leq k+1}$ restricts to a cone on $\alpha^{\leq k}$, hence there is a universal arrow

$$\tilde{e} : \tilde{L} \rightarrow L^{\leq k}.$$

Because $\tilde{\sigma}^{\leq k+1}$ has the limit property even at the height $k+1$, $\tilde{\sigma}^{\leq k+1}$ satisfies $u_2 \cdot u_1 \cdot \tilde{e} = u_3 \cdot u_1 \cdot \tilde{e}$, and thus \tilde{e} factors through e uniquely, as desired. \square

Definition 23. Let $F : C \rightarrow C'$ be a functor. Then F *preserves limits* or is *left exact* if for every functor $\alpha : I \rightarrow C$,

$$F(\lim(\alpha)) \subset \lim(F \circ \alpha).$$

Dually, F *preserves colimits* or is *right exact* if for every functor $\alpha : I \rightarrow C$,

$$F(\text{colim}(\alpha)) \subset \text{colim}(F \circ \alpha).$$

F is said to *create limits* if for every element $\pi \in \lim(F \circ \alpha)$, there exists a unique $\pi' \in \lim(\alpha)$ such that $F(\pi') = \pi$. Dually, F is said to *create colimits* if for every element $\pi \in \text{colim}(F \circ \alpha)$, there exists a unique $\pi' \in \text{colim}(\alpha)$ such that $F(\pi') = \pi$.

For example, the hom functor

$$b \mapsto \text{hom}(-, b)$$

preserves limits. Dually, the contravariant hom functor

$$a \mapsto \text{hom}(a, -)$$

preserves colimits. These functors may be extended to generalized categories [14].

Theorem 5.2. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between generalized categories \mathcal{C} and \mathcal{D} . Then if F has a left adjoint $G : \mathcal{D} \rightarrow \mathcal{C}$, then it is left exact.*

Proof. Like the proof for categories, the proof for generalized categories relies on naturality of the adjoints via the bijection (5). \square

The dual statement to 5.2 is immediate: a functor with a right adjoint is right exact.

6. CONCLUSION

We have carried out an investigation of assumptions about the basic notions in category theory, with the motivation being to find a stable and sufficiently rich generalization. There are numerous advanced notions of category theory that have not yet made an appearance in our development, for example, ends, coends, monads, Kan extensions, to name only a few. Some of these are treated in the generalized setting in [14].

Our investigation has yielded the following observations: There exists a generalization of category theory. More precisely, there exists a theory of functors, natural transformations, adjoint pairs, limits, and colimits for generalized categories. Still more precisely, there are two generalizations that are combined into one larger one: First, by allowing an approximate operation of composition (i.e., proximal categories), and second, by allowing generalized higher cells. We have seen that the structure of limits and natural transformations is similar to the structure as it arises in ordinary one-categories, so that, surprisingly, perhaps, the proximal structure has little effect on aspects of the theory. We have investigated a notion of non-natural transformation suggested by the one-categorical case where naturality is not a necessary assumption, and we have found that the device

of non-natural equivalence is not sufficiently strong. Therefore, we have argued that naturality must be an explicit assumption in the generalized setting. Thus, we have both extended the boundaries of category theory, and made note of some limits to further extensions of the new boundaries we have drawn, which strengthens the case for our particular approach. With the foundations developed here, it is possible to go further.

APPENDIX A. THE 2-CATEGORY OF CATEGORIES

The following is a brief review of perhaps the most important elementary construction in category theory: the strict 2-category of categories.

Let \mathcal{C}, \mathcal{D} be categories. Two natural transformations $\beta : G \Rightarrow H, \alpha : F \Rightarrow G$ between functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$ may be composed via the rule

$$\beta \triangle \alpha(X) := \beta(X) \cdot \alpha(X)$$

where (\cdot) denotes composition in \mathcal{D} . This gives a category $\text{Nat}(\mathcal{C}, \mathcal{D})$. Identities in $\text{Nat}(\mathcal{C}, \mathcal{D})$ are given by $\text{id}_F(X) := \text{id}_X$.

Given natural transformations $\alpha : F \Rightarrow G$ between functors $\mathcal{C} \rightarrow \mathcal{D}$, and $\beta : F' \Rightarrow G'$ between functors $\mathcal{D} \rightarrow \mathcal{E}$, we obtain a well-defined function $\text{Ob}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{E})$ via

$$\beta \star \alpha(X) := \alpha(\hat{\beta}(X)) \cdot \bar{\alpha}(\beta(X)),$$

where hats and bars are used as defined in section 2 below. This can also be written

$$\beta \star \alpha = (\alpha \circ \hat{\beta}) \triangle (\bar{\alpha} \circ \beta)$$

Note that

$$\begin{aligned} \bar{\alpha}(X) &= \overline{\alpha(X)}, \\ \hat{\alpha}(X) &= \widehat{\alpha(X)}. \end{aligned}$$

Proposition A.1 (The Five Facts). *In the notation above, whenever expressions on both sides of the formula are defined, we have:*

- (1) $\beta \star \alpha = (\hat{\alpha} \circ \beta) \triangle (\alpha \circ \bar{\beta})$.
- (2) $\beta \star \alpha$ is a natural transformation $G \circ F \Rightarrow G' \circ F'$.
- (3) $(\gamma \star \beta) \star \alpha = \gamma \star (\beta \star \alpha)$.
- (4) If

$$\begin{aligned} \left. \begin{array}{l} \alpha : F \Rightarrow G \\ \beta : G \Rightarrow H \end{array} \right\} : \mathcal{C} \rightarrow \mathcal{D}, \\ \left. \begin{array}{l} \alpha' : F' \Rightarrow G' \\ \beta' : G' \Rightarrow H' \end{array} \right\} : \mathcal{D} \rightarrow \mathcal{E}, \end{aligned}$$

then

$$(\beta' \triangle \alpha') \star (\beta \triangle \alpha) = (\beta' \star \beta) \triangle (\alpha' \star \alpha).$$

- (5) If id_F^Δ is the identity of F with respect to the product \triangle in $\text{Nat}(\mathcal{C}, \mathcal{D})$, then

$$\alpha \star \text{id}_F^\Delta = \alpha,$$

$$\text{id}_F^\Delta \star \beta = \beta,$$

whenever both sides are defined.

Proof. (1)

$$\begin{aligned}
(\beta \star \alpha)(X) &= (\alpha \circ \hat{\beta}) \triangle (\bar{\alpha} \circ \beta)(X) \\
&= \alpha(\hat{\beta}(X)) \cdot \bar{\alpha}(\beta(X)) \\
&= \alpha(\widehat{\beta(X)}) \cdot F(\beta(X)) \\
&= G(\beta(X)) \cdot \alpha(\overline{\beta(X)}) \\
&= \hat{\alpha}(\beta(X)) \cdot \alpha(\bar{\beta}(X)) \\
&= (\hat{\alpha} \circ \beta) \triangle (\alpha \circ \bar{\beta})(X).
\end{aligned}$$

(2) by Fact 1.

(3) Apply the definition.

(4) by Fact 1 and since $\hat{\alpha} = \bar{\beta}$, $\hat{\alpha}' = \bar{\beta}'$.

(5) direct calculation. □

Remarks.

(1) We may write simply id_F or 1_F in light of Fact (5).

(2) Fact 4 is often referred to as the *interchange law*.

An immediate consequence of the Five Facts is the following: *The category of categories is a strict two-category.* By “the category of categories” is meant the set of small categories, functors, and natural transformations in a fixed universe $\mathcal{U}_{\text{univ}}$.

APPENDIX B. GLOBULAR SETS

It is worthwhile to remark on the relationship between generalized categories and globular sets. Recall that a globular set is a presheaf of shape \mathbb{G} (that is, a functor $\mathbb{G}^{\text{op}} \rightarrow \mathbf{Set}$), where \mathbb{G} is the category of natural numbers $n \geq 0$ together with maps

$$\begin{array}{ccccccc}
0 & \xrightarrow{\sigma_0} & 1 & \xrightarrow{\sigma_1} & 2 & \xrightarrow{\sigma_2} & \dots \\
& & \xrightarrow{\tau_0} & & \xrightarrow{\tau_1} & & \xrightarrow{\tau_2}
\end{array}$$

subject to the relations $\sigma_{i+1} \circ \sigma_i = \tau_{i+1} \sigma_i$, $\tau_{i+1} \circ \tau_i = \sigma_{i+1} \circ \tau_i$, for $i \geq 0$.

Definition 24. Let \mathcal{C} be a generalized category. A k -cell in \mathcal{C} is an element f of \mathcal{C} such that for every k -element sequence \vec{s} of operations \mathbf{s} and \mathbf{t} that satisfy when applied to f ,

- (1) $\mathbf{s}^k f$ and \mathbf{t}^k are objects, and $\mathbf{s}^{k-n} f$ and \mathbf{t}^{k-n} are not objects, for all $0 \leq n \leq k$,
- (2) $\mathbf{s} \mathbf{t} f = \mathbf{s} \mathbf{s} f$ and $\mathbf{t} \mathbf{s} f = \mathbf{t} \mathbf{t} f$,
- (3) $\mathbf{s} f$ and $\mathbf{t} f$ is are $k - 1$ -cells.

For example, in a 1-dimensional category, all elements are 1-cells, and some elements are also 0-cells. An element f of a generalized category \mathcal{C} is *cellular* if f is a k -cell for some $k \geq 1$, and a generalized category \mathcal{C} is *cellular* if every element of \mathcal{C} is cellular.

Proposition B.1. *There is an equivalence (given by a forgetful-free adjunction) between sharp, cellular generalized categories and the category of globular sets.*

Proof. To prove this, we must be sure clarify the statement: when referring to sharp, cellular generalized categories, we refer not to the full subcategory but to the category whose morphisms $F : \mathcal{C} \rightarrow \mathcal{D}$ are subject to the extra condition

- (1) for all $a \in \mathcal{C}$, $\mathbf{s}(F(a)) = F(a)$ implies $\mathbf{s} a = a$.

This says we cannot map k -cells for $k > 0$ to 0-cells. Then let $\dim(a) := \min\{n \mid \mathbf{s}^n a = \mathbf{s}^{n+1} a\}$. Define a mapping

$$\mathcal{C} \mapsto (n \mapsto \{a \in \mathcal{C} \mid \dim a = n\}).$$

to the category of globular sets, for a sharp cellular generalized category \mathcal{C} . This is the desired equivalence. \square

Examples of noncellular generalized categories are abundant, for example arising from the theory of trees and related notions, see for example [6].

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